# UNIVERSITY EXAMINATIONS

## 2022/2023 ACADEMIC YEAR

## SECOND YEAR SECOND SEMESTER

## SUPPLEMENTARY EXAMINATION

## FOR THE DEGREE OF BACHELOR OF SCIENCE MATHEMATICS

**COURSE CODE:** **MAP 223**

COURSE TITLE: **ALGEBRAIC STRUCTURES II**

**DATE**: 9/8/2023 **TIME**: 8:00 AM - 10:00 AM

## INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

## QUESTION ONE COMPULSORY (30 MARKS)

a) Define the following terms
   i. Group (4marks)
   ii. Division algorithm (3marks)
   iii. Field (2marks)

b) Let G be the group which consists of the six matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

With the operation of matrix multiplication, construct the table for this group (10marks)

c) Let n be a fixed integer, and let $H = \{x \in G: x^n = e\}$. Prove that H is a subgroup of G (6marks)

d) Let G be a group and $a, b \in G$. Prove by induction $(bab^{-1})^n = ba^n b^{-1}$ (5marks)

## QUESTION TWO (20 MARKS)

a) Define the following terms
   i. Binary operation (2marks)
   ii. Homomorphism (2marks)
   iii. Subgroup (3marks)

b) If G and H are groups, prove that $G \times H$ is also a group (8marks)

c) Let $\emptyset: G \to G'$ be a homomorphism of groups. Let $e \in G$ be the identity element of G , prove that $\emptyset(e)$ is the identity of G' (5marks)

## QUESTION THREE (20 MARKS)

a) Define the following terms
   i. Group Isomorphism (2marks)
   ii. Generators of a group (2marks)
   iii. Cyclic group (2marks)
   iv. Lagrange's theorem (2marks)

b) State three applications of Symmetries and their groups (3marks)

c) Show that for $n \geq 1$, $8^n - 3^n$ is divisible by 5 for $n \in \mathbb{N}$ by induction (6marks)

d) Solve simultaneously $x^2 a = bxc^{-1}$ $and\ acx = xac$ (3marks)

## QUESTION FOUR (20 MARKS)

a) If G is a group and a,b are elements of G then prove that
   i. $ab = ac$ $implies$ $b = c$ (2marks)
   ii. And $ba = ca$ $implies$ $b = c$ (2marks)

b) Let $H = \{x \in G: x = y^2$ $such\ that$ $y \in G\}$ Prove that H is a subgroup of G (4marks)

c) State residue classes of integers mod 4 (4marks)

d) Let $G$ be the subset of $S4$ consisting of the permutations

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Show that G is a group of permutations, and write its table (8marks)

## QUESTION FIVE (20 MARKS)

a) Let G be a cyclic group of order 9. How many of its elements generate G? (4marks)

b) Prove that the identity element in any group G is unique (3marks)

c) Let $G_1$ and $G_2$ be groups and $\phi: G_1 \times G_2 \rightarrow G_1 \times G_2$. Prove that $\phi$ is an isomorphism (6marks)

d) Prove that if $abc = e$ then $cab = e$ and $bca = e$ (4marks)

e) State the Lagrange's theorem (3marks)