UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

THIRD YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION (SCIENCE)

COURSE CODE: **MAP 324**

COURSE TITLE: GROUP THEORY

DATE: 21/04/23 **TIME:** 9 AM -11 AM

INSTRUCTIONS TO CANDIDATES

Answer Question ONE and Any TWO Questions

TIME: 2 Hours

This Paper Consists of 3 Printed Pages. Please Turn Over.

QUESTION ONE (30MARKS)

a. Define the following
   i. Trivial subgroups (2 marks)
   ii. Proper subgroup (2 marks)
   iii. Group (3 marks)

b. Let G be a group. State three conditions under which a subset H is a subgroup of G (3 marks)

c. Let G be a group and a,b∈ G. Show that $(a.b)^{-1} = b^{-1}a^{-1}$ (4 marks)

d. Let G be a group, suppose x∈ G. Show that x has exactly one inverse $x^{/}$ (5 marks)

e. Show that every permutation can be expressed as a product of transpositions (4 marks)

f. Compose the permutation (1234)∗(13)(24) in cycle notations (2 marks)

g. Represent the permutation (13584)(2967) $\in S_9$ as a product of transpositions (2marks)

h. Describe in detail the word "transposition" (3marks)

QUESTION TWO (20MARKS)

a. Define the following
   i. Isomorphism (2marks)
   ii. Automorphism (2marks)

b. Let $\varphi: G \to H$ be a homomorphism and let e, $e^{/}$ denote the identity elements of G and H respectively. Show that
   i. $\varphi(e) = e^{/}$ (2marks)
   ii. $\varphi(a^{-1}) = \varphi(a)^{-1}$ (2 marks)
   iii. $\varphi(a^{n}) = \varphi(a)^{n}$ for all a∈G, n∈Z (2marks)

c. Show that φ is a monomorphism if and only if ker φ = {e}. (10 marks)

QUESTION THREE (20MARKS)

a. Define the following
   i. Center of a group (2marks)
   ii. Homomorphism (2marks)

b. Let K= Ker (ϕ). Define i: G⁄K ⟶ im (ϕ) where i: gK⟶ ϕ (g). Show that
   i. i is well defined (6 marks)
   ii. i is a homomorphism (3 marks)
   iii. i is surjective (2 marks)
   iv. i is injective (5 marks)

QUESTION FOUR (20MARKS)

a. Define the following
   i. Right Coset (2marks)
   ii. An index of a subgroup (2 marks)

b. Let H be the subgroup of S_3 defined by the permutations $\{(1), (123), (132)\}$. Find the left cosets of H. (8 marks)

c. Let H be a subgroup of a group G. Show that the group G is the disjoint union of the left cosets of H in G. (8marks)

QUESTION FIVE (20MARKS)

a. Define the following
   i. Normal subgroup (2marks)
   ii. Factor group (2marks)

b. Let G be a group and N a subgroup of G. Show that the following statements are equivalent ;
   i. The subgroup N is normal in G
   ii. For all $g \in G$, $gNg^{-1} \subset N$
   iii. For all $g \in G$, $gNg^{-1} = N$ (11marks)

c. Let N be a normal subgroup of a group G. Show that the cosets of N in G forms a group G/N of order [G: N] (5 marks)