UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION

AND BACHELOR OF SCIENCE

COURSE CODE: MAP 422

COURSE TITLE: DIFFERENTIAL TOPOLOGY

DATE: 11/08/2023 **TIME**: 8:00 AM - 10:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of 3 Printed Pages. Please Turn Over.

SECTION A [30 MARKS] Compulsory

QUESTION ONE.

(a) Using a unit circle as an example, define the terms Manifold. [6 MARKS]

(b) Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ be open sets. State the condition under which a map $\psi : U \to V$ is smooth. Show that the map $\psi : \mathbb{R}^2 \to \mathbb{R}, (\ x,\ \ y,\ ) \mapsto (\ x^2 + 2xy - y^2\ )$ is smooth at a point $(-1, 3) \in \mathbb{R}^2$. [6 MARKS]

(c) Let $p = (a, b) \in S^1$ be a point with $b > 0$. A local parametrization around p with $\phi(0) = p$ is given by

$$\phi : (-\epsilon, \epsilon) \to S^1, t \mapsto (t + a, \sqrt{1 - (t + a)^2})$$

for some small enough real number $\varepsilon > 0$. Find $d\phi_t$. [6 MARKS]

(d) Define a submersion map. Show that the linear map

$$f : \mathbb{R}^4 \to \mathbb{R}, (x, y, w, z) \mapsto x^3 + y^2 + w + z^4$$

is a submersion for all points in $(x, y, w, z) \in \mathbb{R}^4$. [6 MARKS]

(e) . Let $f : S^3 \to S^2$ defined by

$$f : S^3 \to S^2, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1x_3 + 2x_2x_4 \\ 2x_2x_3 - 2x_1x_4 \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}$$

and

$$g : S^3 \to S^4, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \\ \sqrt{1 - (x^2 + y^2 + z^2)} \end{pmatrix}$$

. Compute the composite map $f \circ g$ [6 MARKS]

QUESTION TWO.[20 Marks]

(a). Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the unit circle. Show that S^1 is a 1-dimensional manifold. [10 MARKS]

(b). Let $\psi_N : \mathbb{R}^2 \to S^2$ define a stereographic projection

$$\psi_N(x, y) = \frac{1}{1 + x^2 + y^2}(2x, 2y, x^2 + y^2 - 1).$$

Find the derivative space $d(\psi_N)$ at (x, y), hence that $d(\psi_N)$ is orthogonal to the vector $\psi_N(x, y)$ [10 MARKS]

QUESTION 3 [20 Marks].

(a). State and prove The Inverse Function. Theorem [12 MARKS]

(b). Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the a smooth map $f : (s, t) \mapsto ((1 + 2\cos s)\cos t, (1 + 2\cos s)\sin t, 2\sin s)$. Using the point $(\frac{\pi}{4}, \frac{\pi}{6})$ show that f is an immersion. [8 MARKS]

QUESTION 4 [20 Marks].

(a). Using Jacobi matrix, show that the linear map $f : (-\pi, \pi) \to \mathbb{R}^2, t \mapsto (\sin 2t, \sin t)$ is smooth, one-to-one and an immersion. [6 MARKS]

(b). For a smooth map of manifolds $f : X \to Y$. Define a regular and a critical value. Show that

$$f : \mathbb{R}^4 \to \mathbb{R}, (x, y, w, z) \mapsto x + y^2 + w^3 - z^4$$

has a regular point $(1, 1, -1, 1) \in \mathbb{R}^4$ [7 MARKS]

(c). Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + xa_1 + x_0$ be a polynomial with real coefficients.Prove that a smooth map $\psi : \mathbb{R} \to \mathbb{R}^2$ defined by $\psi(x) = (x, \psi(x))$ is transverse to the x-axis Z if and only if all zeros of $p(x)$ are simple. [7 MARKS]

QUESTION 5 [20 Marks].

(a). Prove Brouwer Fixed-Point Theorem for continuous maps that state "Every continuous map $F : D^n \to D^n$ has a fixed point". [10 MARKS]

(b). Let X be a smooth manifold with or without boundary and let $Y \subset \mathbb{R}^n$ be a smooth manifold without boundary. Let $f : X \to Y$ be a continuous homotopic map. Show that if f is a smooth map on a closed subset $A \subset X$ then the homotopy can be chosen relative to A. [10 MARKS]