SPECIAL/SUPPLEMENTARY UNIVERSITY EXAMINATIONS

ACADEMIC YEAR 2022/2023

THIRD YEAR SECOND SEMESTER EXAMINATIONS

BACHELOR OF SCIENCE

COURSE CODE: SPC 323

COURSE TITLE: MATHEMATICAL PHYSICS II

DATE: 15/8/2023 **TIME: 8:00-10:00AM**

INSTRUCTIONS TO CANDIDATES

Answer question ONE and any TWO of the remaining.

Time: 2 hours

QUESTION ONE (30 MARKS)

a) Find the Laurent series for $f(z)=\frac{z}{z^2+1}$ (4marks)

b) The function $f(z)=\frac{1}{z(z-1)}$ has isolated singularities at z=0 and z=1. Show that both are simple poles (4 marks)

c) Show that if a is a constant, then $u(x,y)=\sin(at)\cos(x)$ is a solution to (4 marks)

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

d) Prove that the Legendre polynomials satisfy the following: (4 marks)

$$(2n+1)P_n(x)=P'_{n+1}(x)-P'_{n-1}(x)$$

e) Show the steps used to obtain these Laplace identities:

$$\textbf{(a)}\ \mathcal{L}(e^{at}\cos bt)=\frac{s-a}{(s-a)^2+b^2}$$

(2 marks)

$$\textbf{(b)}\ \mathcal{L}(te^{at}\cos bt)=\frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$$

(4 marks)

f) Find the Laurent series about the singularity for the function: (4 marks)

$$\frac{e^x}{(z-2)^2}$$

g) Use the Fourier integral to prove that (4 marks)

$$\int_0^\infty \frac{\cos ax\,dx}{1+a^2}=\frac{\pi}{2}e^{-x}$$

QUESTION TWO (20 MARKS)

a) For *k* constant, find the separated solution to the Heat Equation

$$\frac{\partial u}{\partial t}=k\frac{\partial u^2}{\partial x^2}$$

(10 marks)

b) Find the separated solution to $F_z+2xF_y=0$ (10 marks)

QUESTION THREE (20 MARKS)

Let $f(x)$ be a function of period 2Π such that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\Pi < x < 2\Pi$ (2 marks)

b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\Pi$ is (9 marks)

$$\frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right] - \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right]$$

c) By giving appropriate values to x show that: (9 marks)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots\dots \quad \text{and} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\dots\dots$$

QUESTION FOUR (20 MARKS)

a) State the transformation properties of tensors T_α and $S^{\alpha\beta}$. Obtain the transformation properties of $T_\alpha S^{\alpha\beta}$. Explain your result. (10 marks)

b) The double summation of $K_{ij} A_i B_j$ is invariant for any two vectors A_i and B_j. Prove that K_{ij} is a second order tensor. (10 marks)

QUESTION FIVE (20 MARKS)

a) For Legendre polynomials $P_l(x)$ the generating function is given by:

$$T(x,s) = (1 - 2sx + s^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)s^l, \; s < 1$$

Use the generating function to show:

(i) $(l+1)P_{l+1} = (2l+1)xP_l - lP_{l-1}$ (5 marks)

(ii) $P_l(x) + 2xP_l'(x) = P_{l+1}'(x) + P_{l-1}'(x)$ (5 marks)

b) Show that the Legendre polynomials have the property (10 marks)

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = \frac{2}{2n+1}, \text{ if } m = n$$
$$= 0, \text{ if } m \neq n$$