UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

THIRD YEAR SECOND SEMESTER

SPECIAL/SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE AND BACHELOR OF EDUCATION

COURSE CODE: **STA 325**

COURSE TITLE: BIVARIATE PROBABILITY DISTRIBUTION

DATE: 15/8/2023 **TIME:** 11:00 A.M - 1:00 P.M

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE COMPULSORY (30 MARKS)

a) The joint density function of two continuous random variables X and Y is

$$f(x,y)=\begin{cases} Cxy & ; \ 0<x<4; \ \ 1<y<5 \\ 0 & ; \ otherwise \end{cases}$$

Find:

i. value of C that makes $f(x,y)$ a probability function. (3 marks)

ii. $P(X<2, 0<Y<3)$ (3 marks)

iii. $P(X>Y)$ (4 marks)

b) Let the joint p.d.f. of $X \ and \ Y$ be

$$f(x,y)=\begin{cases} \frac{1}{21}(x+y); & x=1,2,3; \ y=1,2 \\ 0 & ; \ otherwise \end{cases}$$

Find the

i. $P(Y=1)$ (2 marks)

ii. marginal probability density function of X and Y (2 marks)

iii. conditional density function of $P(Y \mid X=x)$, hence $P(Y \mid X=4)$ (3 marks)

iv. conditional mean of $P(Y \mid X=2)$, hence $P(Y \mid X=4)$ (3 marks)

c) The discrete random variable X takes the value 0 with probability 0.2 and the value 1 with probability 0.8. The discrete random variable Y takes the value 0 with probability 0.4 and the value 1 with probability 0.6. If $X \ and \ Y$ are correlated and $P(X=1, Y=1)=0.5$.

i. Produce a table that shows all the values of the joint probability distribution of (X,Y) (2 marks)

ii. Find the CDF of X, hence use it to find $F\left(\frac{1}{2},1\right)$ (3 marks)

iii. Show that the correlation between X and Y is $Corr(X,Y)=\rho_{xy}=0.102062072$ (5 marks)

QUESTION TWO (20 MARKS)

a) The continuous random variables $X \ and \ Y$ are jointly distributed with joint probability density function $f(x,y)=\begin{cases} kx^2 & ; \ 0<x<1, 0<y<x \\ 0 & ; \ otherwise \end{cases}$

where k is a constant.

i. Show that $k=4$. (3 marks)

ii. Find the marginal probability density function of Y (2 marks)

iii. Find the conditional density function of $X \ given Y=y$ (2 marks)

iv. Find the conditional CDF of $X \ given Y=y$ (2 marks)

v. Hence or otherwise, evaluate $P\left[X<\tfrac{1}{2} \middle/ X+Y<1\right]$ (5 marks)

vi. Are random variables $X \ and \ Y$ independent? (3 marks)

b) Suppose that X and Y are random variables such that $Var(X)=9$, $Var(Y)=4$ and $Corr(X,Y)=\rho_{xy}=\frac{1}{4}$. Determine the $Var(X-3Y-4)$ (3 marks)

QUESTION THREE (20 MARKS)

a) The joint distribution function is given as

$$f(x,y)=\begin{cases}\frac{1}{15}(x+y) & 0\le x\le 2\ \ ,0\le y\le 3\\ 0 & otherwise\end{cases}$$

Determine the value of $E(3X-4Y-2)$ (4 marks)

b) Suppose that X and Y are the heights and weights of a certain animal and have a bivariate continuous probability density function given by

$$f(x,y)=\begin{cases}e^{-y} & ;0<x<y<\infty\\ 0 & ;o.w.\end{cases}$$

i. Show that, the joint moment generating function of X and Y is $M(t_1,t_2)=\frac{1}{(1-t_2)(1-t_1-t_2)};t_1+t_2<1,\ t_2<1$ (4 marks)

Hence, use the m.g.f. obtained in (i). above to find;

ii. $E(X)$ and $Var(X)$ (3 marks)

iii. $E(Y)$ and $Var(Y)$ (3 marks)

iv. $Cov(XY)=\sigma_{xy}$ and correlation coefficient $Cor(XY)=\rho_{xy}$ (4 marks)

v. Giving a reason(s), are random variables X and Y independent? (2 marks)

QUESTION FOUR (20 MARKS)

a) The trinomial of two random variables X and Y is given by

$$f(x,y)=\begin{cases}\frac{n!}{x!y!(n-x-y)!}p^x q^y(1-p-q)^{n-x-y} & ;\ x,y=0,1,2,\dots,n,\ x+y\ge n,0\le p,0\le q,q+p\le 1\\ 0 & ;\ otherwise\end{cases}$$

i) Find the discrete density function of y (3 marks)

ii) Find the conditional distribution of X given $Y=y$ and obtain its expected value (3 marks)

b) Suppose W and V are independent random variables where W has standard normal distribution and V has χ_r^2 distribution. Let $T=\frac{W}{\sqrt{\frac{V}{r}}}$ and $U=V$.

i. Find the joint probability density function of T and U, i.e. $f(t,u)$.

ii. Hence, use it to find the probability density function of T, i.e. $f(t)$

iii. Find $E[T]$ and $Var[T]$. (14 marks)

QUESTION FIVE (20 MARKS)

a) Suppose that X and Y have the bivariate normal density with mean vector and covariance matrix given by $\mu=\begin{bmatrix}10\\9\end{bmatrix}$ and $\sum=\begin{bmatrix}16 & -0.5\\ -0.5 & 9\end{bmatrix}$ respectively.

Let $Z=(2X-Y)$.

i) Determine $E(Z)$ and $Var(Z)$ (4 marks)

ii) Determine $P(2 < Z < 25)$ (3 marks)

iii) Find the correlation between X and Y. (2 marks)

iv) Find the conditional distribution of X given Y, i.e. $f(X \mid Y = y) = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y - \mu_2)$

(2 marks)

b) In a large shipment of parts, 1% of the parts do not conform to specifications. The supplier inspects a random sample of 30 parts, and the random variable X denotes the number of parts in the sample that do not conform to specifications. The purchaser inspects another random sample of 20 parts, and the random variable Y denotes the number of parts in this sample that do not conform to specifications. What is the probability that $X \leq 1$ and $Y \leq 1$, i.e. $P(X \leq 1, Y \leq 1)$ assuming that X and Y are independent? (4 marks)

c) Let X and Y be independent normal random variables such that $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$. Show that a random variable $V = AX + BY$, is distributed as $V \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$, where A and B are constants. (5 marks)