UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION AND BACHELOR OF SCIENCE

COURSE CODE: **STA 424**

COURSE TITLE: STOCHASTIC PROCESSES II

DATE: 09/08/2023 **TIME:** 2:00 PM – 4:00 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of ***3 Printed Pages****. Please Turn Over.*

QUESTION ONE: (30 Marks) (COMPULSORY)

a) Define the following terms

i. Absorbing state [1mk]

ii. Irreducible markov chains [1mk]

iii. Period of a state of markov chains [1mk]

b) Consider a series of Bernoulli trials with probability of success p. Suppose that X denote the number of failures preceding the first success and Y the number of failures following the first success and preceding the second success. The joint pdf of X and Y is given by

$P_{ij} = pr\{x = j, y = k\} = q^{j+k}p^2 \quad j, k = 0, 1, 2, 3, \ldots$ Obtain the;

i. Bivariate probability generating function of X and Y [2mks]

ii. marginal probability generating function of X and Y [2mks]

iii. mean and variance of X [1mk]

iv. mean and variance of Y [1mk]

c) Given that random variable X have probability density function $pr(X = k) = p_k, \quad k = 0, 1, 2, 3, \ldots.$ with probability generating function $P(S) = \sum_{i=1}^{\infty} p_k s^k$ and $q_k = p_k(X = k) = p_{k+1} + p_{k+2} + p_{k+3} + \cdots$ with generating function $\phi(s) = \sum_{i=1}^{\infty} q_k s^k$

Show that $(1 - s)\phi(s) = 1 - p(s)$ and that $E(X) = \phi(1)$ [4mks]

d) Find the generating function for the sequence $a_k = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ for fixed n [2mks]

e) Let X have the distribution of the geometric distribution of the form

$Prob\ (X = k) = p_k = q^{k-1}\,p\ , \quad k = 1, 2, 3, \ldots$

Obtain the probability generating function and hence find its mean and variance [6mks]

f) Classify the state of the following stochastic markov chain

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 \end{matrix} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

[9mks]

QUESTION TWO: (20 Marks)

a) Let X have a Bernoulli distribution with parameters p and q given by

$P_r(X = k) = P_k = P^k q^{1-k}, \quad q = 1 - p, \; k = 0, 1$

Obtain the probability generating function of X and hence find its mean and variance. [6mks]

b) The difference – differential equation for pure birth process are

$P'_n(t) = \lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t), \quad n \geq 1$ and

$P'_0(t) = -\lambda_0 p_0(t), \; n = 0.$

Obtain $P_n(t)$ for a non – stationary pure birth process (Poisson process) with $\lambda_n = \lambda$ given that

$$P_0(t) = \begin{cases} 1 & for\; n = 0 \\ 0 & otherwise \end{cases}$$

Hence obtain its mean and variance [14mks]

QUESTION THREE: (20 Marks)

a) Let X have a Poisson distribution with parameter λ i.e.

$Prob\,(X = k) = p_k = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, 3, \dots$

Obtain the probability generating function of X and hence obtain its mean and variance [6mks]

b) Using Feller's method, find the mean and variance of the difference – differential equation

$P'_n(t) = -n(\lambda + \mu)p_n(t) + (n-1)\lambda p_{n-1}(t) + \mu(n+1)p_{n+1}(t), \; n \geq 1$ given

$m_1(t) = \sum_{n=0}^{\infty} n p_n(t)$, $m_2(t) = \sum_{n=0}^{\infty} n^2 p_n(t)$ and

$m_3(t) = \sum_{n=0}^{\infty} n^3 p_n(t)$ conditioned on $p_1(0) = 0, \; p_n(0) = 0, \; n \neq 0$

[14mks]

QUESTION FOUR: (20 Marks)

a) Define the following terms

i. Transient state [1mk]

ii. Ergodic state [1mk]

iii. Recurrent state [1mk]

b) Classify the state of the following transitional matrix of the markov chains

$$\begin{array}{c} \\ E_1 \\ E_1 \\ E_1 \\ \vdots \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & \cdots \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdots \\ 1/2 & 0 & 0 & 1/2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/2 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \end{array}$$

[17mks]

QUESTION FIVE: (20 Marks)

a) Let X have a binomial distribution with parameter n and p i.e.

$$Prob\ (X = k) = p_k = \binom{n}{k} p^k q^{n-k}, \qquad k = 0, 1, 2, 3, \dots, n$$

Obtain the probability generating function of X and hence find its mean and variance. [6mks]

b) The difference – differential equation for the simple birth – death processes are

$P'_n(t) = -n(\lambda + \mu)p_n(t) + (n-1)\lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t),\ n \geq 1$ and

$P'_0(t) = \mu p_1(t),\ n = 0$

Obtain $P_n(t)$ for a simple Birth – Death process with $\lambda_n = n\lambda$ and $\mu_n = n\mu$ given that $P_n(0) = \begin{cases} 1 & for\ n = 1 \\ 0 & for\ n = 0 \end{cases}$ [14mks]